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Massive scalar field in a one-dimensional oscillating region

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Abstract
We study the classical scalar massive field satisfying the Klein–Gordon equation in a finite one-dimensional space interval of periodically varying length with Dirichlet boundary conditions. For a sufficiently small mass, the energy can exponentially grow with time under the same conditions as for the massless case. The proofs are based on estimates of exactly given mass-induced corrections to the massless case.

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1. Introduction

The classical massless scalar field satisfying the d'Alembert equation in 1+1-dimensional space–time restricted to a finite space interval, with one end-point fixed and the other end-point periodically oscillating, has been studied in several papers [1–7]. At the end-points, Dirichlet boundary conditions are assumed. The results for Neumann boundary conditions are also known [5] but the physical condition on the moving boundary, obtained from the static Neumann boundary condition by Lorentz transformation, is different and the results for this condition are more similar to those for the Dirichlet boundary condition. Various regimes of the energy time evolution are possible; the energy can be unlimited or bounded, with limit or without limit at time infinity. In particular, there are cases where energy $E(t)$ exponentially grows with time t in the sense that

$$A\,\mathrm{e}^{\gamma t} \leqslant E(t) \leqslant B\,\mathrm{e}^{\gamma t}$$

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for all $t > 0$ with some $A, B, \gamma > 0$ (the energy is not monotone in time for a periodic wall motion, of course).

A similar model in quantum field theory was also studied in [8] and references therein.

In this paper, we consider the classical massive scalar field satisfying the Klein–Gordon equation with Dirichlet boundary conditions at the end-points of a finite one-dimensional space interval, one end-point periodically moving. We prove that the simplest sufficient conditions for the exponential growth of the energy are the same as for the d'Alembert equation, provided that the mass is small enough. This shows that the mass can be considered as a small perturbation.

2. The model

Let φ be a real function defined on the space–time region

$$\Omega = \{(t, x) \in \mathbb{R}^2 | 0 \leqslant x \leqslant a(t), t \geqslant 0\} \tag{1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a strictly positive periodic C^2-function with a period $T > 0$. We assume that $|\dot{a}(t)| < 1$ (subluminal velocity of the end-point motion). We restrict ourselves to the fields φ which are in $C^2(\overset{\circ}{\Omega}) \cap C^1(\Omega)$ and assume that the Klein–Gordon equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + m^2 \varphi = 0 \tag{2}$$

is satisfied in the interior $\overset{\circ}{\Omega}$. The constant mass m is non-negative. On the boundary, the conditions

$$\varphi(0, x) = \varphi_0(x) \qquad \frac{\partial \varphi(0, x)}{\partial t} = \varphi_1(x) \qquad (0 < x < a(0)) \tag{3}$$

$$\varphi(t, 0) = \varphi(t, a(t)) = 0 \qquad (t \geqslant 0) \tag{4}$$

are required.

Let us define

$$h = \mathrm{Id} - a \qquad k = \mathrm{Id} + a \qquad F = k \circ h^{-1} \tag{5}$$

as in [5]. Then $h, k, F : \mathbb{R} \to \mathbb{R}$ are increasing C^2-functions. The identities

$$F = \mathrm{Id} + 2a \circ h^{-1} = 2h^{-1} - \mathrm{Id} \qquad F^{-1} = \mathrm{Id} - 2a \circ k^{-1} = 2k^{-1} - \mathrm{Id} \tag{6}$$

are useful in some calculations.

We rewrite the problem in new variables

$$\xi = t + x \qquad \eta = t - x. \tag{7}$$

It is not difficult to see that Ω is transformed into the set $\tilde{\Omega}$ described by inequalities

$$|\eta| \leqslant \xi \leqslant F(\eta) \qquad \eta \geqslant -a(0) \tag{8}$$

or equivalently

$$\max\{-\xi, F^{-1}(\xi)\} \leqslant \eta \leqslant \xi \qquad \xi \geqslant 0 \tag{9}$$

the last inequalities in equations (8) and (9) following automatically from the first ones in fact. The inequalities describing $\tilde{\Omega}$ can also be written distinguishing two cases

$$0 \leqslant \xi \leqslant a(0) \qquad -\xi \leqslant \eta \leqslant \xi \tag{10}$$

$$\xi \geqslant a(0) \qquad F^{-1}(\xi) \leqslant \eta \leqslant \xi. \tag{11}$$

The transformed field

$$\tilde{\varphi}(\xi, \eta) = \varphi(t, x) \tag{12}$$

satisfies the equation

$$\frac{\partial^2 \tilde{\varphi}}{\partial \xi \partial \eta} = -\frac{m^2}{4}\tilde{\varphi} \tag{13}$$

in $\mathring{\tilde{\Omega}}$ with boundary conditions given by the transformation of equations (3) and (4).

The energy of the field

$$\begin{aligned} E_m(t) &= \frac{1}{2}\int_0^{a(t)} \left[\left(\frac{\partial \varphi(t,x)}{\partial t}\right)^2 + \left(\frac{\partial \varphi(t,x)}{\partial x}\right)^2 + m^2 \varphi(t,x)^2\right] \mathrm{d}x \\ &= \int_0^{a(t)} \left[\left(\frac{\partial \tilde{\varphi}(\xi,\eta)}{\partial \xi}\right)^2 + \left(\frac{\partial \tilde{\varphi}(\xi,\eta)}{\partial \eta}\right)^2 + \frac{1}{2} m^2 \tilde{\varphi}(\xi,\eta)^2\right] \mathrm{d}x. \end{aligned} \tag{14}$$

3. Solution of inhomogeneous equation

As a preliminary step, let us consider the equation

$$\frac{\partial^2 \tilde{\varphi}}{\partial \xi \partial \eta} = f(\xi, \eta) \tag{15}$$

with given function $f \in \mathrm{C}^1(\tilde{\Omega})$ for an unknown function $\tilde{\varphi}$ satisfying the same boundary conditions as the required solution of the Klein–Gordon equation. We put $f = -\frac{1}{4}m^2\tilde{\varphi}$ after some calculations. The results of this and the next section do not require the periodicity of function a so it is not assumed here.

We put equation (15) into an integral form, integrating twice. The integration bounds must be taken in such a way that the integration domain is included in $\tilde{\Omega}$. A possible choice is

$$\frac{\partial \tilde{\varphi}(\xi, \eta)}{\partial \xi} = -\int_\eta^\xi f(\xi, z)\,\mathrm{d}z + H_1(\xi) \tag{16}$$

with an arbitrary function H_1 (which must be in C^1 if $\tilde{\varphi}$ is in C^2, of course). It is clear from equation (9) that if $(\xi, \eta) \in \tilde{\Omega}$ then $(\xi, z) \in \tilde{\Omega}$ for all $\eta \leqslant z \leqslant \xi$ so the choice of integration bounds is possible.

Similarly

$$\tilde{\varphi}(\xi, \eta) = \int_{|\eta|}^{\xi} \frac{\partial \tilde{\varphi}(y, \eta)}{\partial y}\,\mathrm{d}y + G_1(\eta) \tag{17}$$

taking into account equation (8). Here, G_1 is again an arbitrary function with a continuous second derivative (possibly with the exception of the point $\eta = 0$). The last two formulae give

$$\tilde{\varphi}(\xi, \eta) = -\int_{|\eta|}^{\xi} \mathrm{d}y \int_\eta^y \mathrm{d}z\, f(y, z) + H(\xi) + G(\eta) \tag{18}$$

where H and G are up to now arbitrary functions (such that $\tilde{\varphi}$ is in C^2) which has to be determined from the boundary conditions and Cauchy data.

For $t = 0$, i.e. $\xi = -\eta = x \in [0, a(0)]$, equations (3) read

$$\varphi_0(\xi) = H(\xi) + G(-\xi) \tag{19}$$

$$\varphi_1(\xi) = -2\int_{-\xi}^{\xi} f(\xi, z)\,\mathrm{d}z + H'(\xi) + G'(-\xi). \tag{20}$$

The Dirichlet boundary condition at $x = 0$, i.e. $\xi = \eta = t \geqslant 0$, gives

$$H(\xi) = -G(\xi) \tag{21}$$

and that at $x = a(t)$, i.e. $\xi = F(\eta)$, $\eta \geqslant -a(0)$, reads

$$0 = -\int_{|\eta|}^{F(\eta)} \mathrm{d}y \int_{\eta}^{y} \mathrm{d}z\, f(y, z) + H(F(\eta)) + G(\eta). \tag{22}$$

By equation (21) we can use function G only

$$\tilde{\varphi}(\xi, \eta) = -\int_{|\eta|}^{\xi} \mathrm{d}y \int_{\eta}^{y} \mathrm{d}z\, f(y, z) + G(\eta) - G(\xi). \tag{23}$$

Equations (19) and (20) give

$$G(\eta) = -\frac{1}{2}\varphi_0(|\eta|)\mathrm{sgn}(\eta) - \frac{1}{2}\int_0^{|\eta|} \varphi_1(x)\,\mathrm{d}x - \int_0^{|\eta|} \mathrm{d}y \int_{-y}^{y} \mathrm{d}z\, f(y, z) + c \tag{24}$$

for $\eta \in [-a(0), a(0)]$. Here, c is an arbitrary constant which we can choose as zero since $\tilde{\varphi}$ is independent of it. The relation (22) gives a prolongation formula for G

$$G(F(\eta)) = G(\eta) - \int_{|\eta|}^{F(\eta)} \mathrm{d}y \int_{\eta}^{y} \mathrm{d}z\, f(y, z) \tag{25}$$

for $\eta \geqslant -a(0)$. Using equations (23)–(25), $\tilde{\varphi}$ is determined in the whole domain $\tilde{\Omega}$.

It remains to prove that the above relations really determine a C^2-function $\tilde{\varphi}$. The only points where the continuity of $\tilde{\varphi}$ and its derivatives require a special check are $\eta = 0$, $\eta = a(0)$ (or $\xi = a(0)$) and their images by the function F (where the continuity then follows automatically). This can be done by a straightforward but a slightly tedious calculation which reveals sufficient and necessary conditions on the Cauchy data φ_0, φ_1 and the right-hand side function f. The necessity of these conditions can also be easily seen from the boundary conditions and their derivatives. It is seen that G and G' are continuous in $[-a(0), \infty)$ while G'' is discontinuous at $\eta = 0$ and continuous in $[-a(0), 0) \cup (0, \infty)$. We summarize the obtained consistency conditions in the following proposition. The derivatives in closed sets are considered as derivatives with respect to these sets here.

Proposition 1. *Let* $a \in \mathrm{C}^2([0, \infty))$, $\inf a > 0$, $|a'| < 1$, $f \in \mathrm{C}^1(\tilde{\Omega})$, $\varphi_0 \in \mathrm{C}^2([0, a(0)])$, $\varphi_1 \in \mathrm{C}^1([0, a(0)])$ *and the following relations are satisfied:*

$$\varphi_0(0) = 0 \qquad \varphi_0(a(0)) = 0 \tag{26}$$

$$\varphi_1(0) = 0 \qquad \varphi_1(a(0)) + a'(0)\varphi_0'(a(0)) = 0 \tag{27}$$

$$\varphi_0''(0) = -4f(0, 0) \tag{28}$$

$$(1 + a'(0)^2)\varphi_0''(a(0)) + a''(0)\varphi_0'(a(0)) + 2a'(0)\varphi_1'(a(0)) = -4f(a(0), -a(0)). \tag{29}$$

Then there exists a unique $\tilde{\varphi} \in \mathrm{C}^2(\tilde{\Omega})$ *satisfying equation (15) in* $\tilde{\Omega}$ *and boundary conditions corresponding to the transformed relations (3) and (4). The solution* $\tilde{\varphi}$ *is given by equations (23)–(25).*

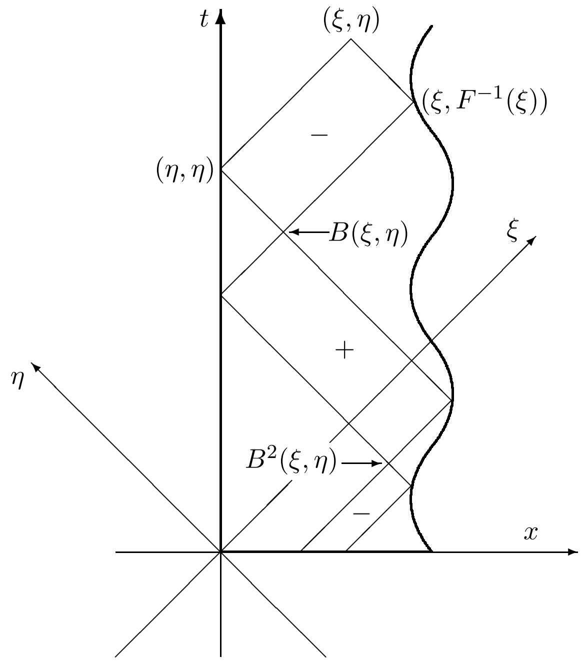

Figure 1. The set $M(\xi, \eta)$ bounded by backward characteristics and the signs of function $\vartheta(\xi, \eta, y, z)$. The coordinates of the points are indicated in variables ξ and η. The number $N(\xi, \eta) = 2$ for the displayed case.

4. Existence and unicity of the solution

We use the results of the previous section to write down the integral form of the Klein–Gordon equation in our case. Let us start with some notations. For $(\xi, \eta) \in \tilde{\Omega}$ let us denote the corresponding time t as

$$T(\xi, \eta) = \frac{\xi + \eta}{2} \tag{30}$$

$$Q(\xi, \eta) = \{(y, z) \in \mathbb{R}^2 | \eta \leqslant y \leqslant \xi,\ F^{-1}(\xi) \leqslant z \leqslant \eta\} = [\eta, \xi] \times [F^{-1}(\xi), \eta] \tag{31}$$

the rectangle bounded by backward characteristics starting from (ξ, η) (see figure 1) and

$$B(\xi, \eta) = (\eta, F^{-1}(\xi)) \tag{32}$$

its lowest vertex. It is easy to verify that the written formulae correspond to figure 1. The following trivial facts are easily seen.

Lemma 1. *Let* $(\xi, \eta) \in \tilde{\Omega}$. *Then*

$$(y, z) \in Q(\xi, \eta) \quad \Longrightarrow \quad T(B(\xi, \eta)) \leqslant T(y, z) \leqslant T(\xi, \eta) \tag{33}$$

$$Q(\xi, \eta) \subset \tilde{\Omega} \quad \Longleftrightarrow \quad B(\xi, \eta) \in \tilde{\Omega} \quad \Longleftrightarrow \quad T(B(\xi, \eta)) \geqslant 0 \tag{34}$$

$$\begin{aligned} Q(\xi, \eta) \cap \tilde{\Omega} &= \{(y, z) \in Q(\xi, \eta) | T(y, z) \geqslant 0\} \\ &= \{(y, z) \in \mathbb{R}^2 | |\eta| \leqslant y \leqslant \xi, \max(F^{-1}(\xi), -y) \leqslant z \leqslant \eta\} \end{aligned} \tag{35}$$

$$T(\xi, \eta) - T(B(\xi, \eta)) = a \circ k^{-1}(\xi) \geqslant \inf a. \tag{36}$$

Lemma 2.

(a) Let $\tilde{\varphi} \in C^2(\tilde{\Omega})$ satisfy equation (13) and the Dirichlet boundary conditions corresponding to equation (4). Then

$$\tilde{\varphi}(\xi, \eta) = -\tilde{\varphi}(B(\xi, \eta)) - \frac{m^2}{4} \int_{Q(\xi,\eta)} \tilde{\varphi}(y, z)\, \mathrm{d}y\, \mathrm{d}z \tag{37}$$

for every $(\xi, \eta) \in \tilde{\Omega}$ such that $T(\xi, \eta) \geqslant 0$.

(b) Let $(\xi_0, \eta_0) \in \tilde{\Omega}$ be such that $T(B(\xi_0, \eta_0)) > 0$ and $\tilde{\varphi} \in C^0(\tilde{\Omega})$ has continuous first derivatives in a neighbourhood of the boundary $\partial Q(\xi_0, \eta_0)$. If $\tilde{\varphi}$ is of class C^2 in a neighbourhood of point $B(\xi_0, \eta_0)$, satisfies equation (13) there and satisfies equation (37) for (ξ, η) in a neighbourhood of (ξ_0, η_0), then $\tilde{\varphi}$ is of class C^2 in a neighbourhood of point (ξ_0, η_0) and satisfies equation (13) there.

Proof. Notice that $Q(\xi, \eta) \subset \tilde{\Omega}$ for $T(B(\xi, \eta)) \geqslant 0$ according to equation (34). For statement (a), integrate equation (13) over (31) and use the Dirichlet boundary conditions at the vertices $(\xi_1 = \eta, \eta)$ and $(\xi, \eta_1 = F^{-1}(\xi))$ (notation as in figure 1). To see statement (b), use equations (31) and (32) and differentiate equation (37). In particular,

$$\frac{\partial^2 \tilde{\varphi}(\xi, \eta)}{\partial\xi \partial\eta} = -\left[\frac{\partial^2 \tilde{\varphi}}{\partial\xi \partial\eta} + \frac{m^2}{4}\tilde{\varphi}\right]_{|B(\xi,\eta)} \frac{\mathrm{d}}{\mathrm{d}\xi} F^{-1}(\xi) - \frac{m^2}{4}\tilde{\varphi}(\xi, \eta).$$

□

Remark. For $m = 0$ the lemma is a special case of the well-known relation of four values at the vertices of a rectangle bounded by characteristics for the solution of the wave equation (see, for example, equation (1.22) of chapter 8 in [9]).

Lemma 3. *Let $(\xi, \eta) \in \tilde{\Omega}$ be such that $T(B(\xi, \eta)) \leqslant 0, \tilde{\varphi}$ satisfies equation (13) with boundary conditions corresponding to equations (3) and (4) and $\tilde{\varphi}^{(0)}$ satisfies equation (13) with m replaced by zero and the same boundary conditions (3) and (4) as $\tilde{\varphi}$. Then*

$$\tilde{\varphi}(\xi, \eta) = \tilde{\varphi}^{(0)}(\xi, \eta) - \frac{m^2}{4} \int_{Q(\xi,\eta)\cap\tilde{\Omega}} \tilde{\varphi}(y, z)\, \mathrm{d}y\, \mathrm{d}z. \tag{38}$$

If $\tilde{\varphi}^{(0)} \in C^2(\tilde{\Omega})$ and $\tilde{\varphi} \in C^0(\tilde{\Omega})$ satisfy the relation (38) then $\tilde{\varphi}$ satisfies the boundary conditions corresponding to equations (3), (4) and (13). Furthermore, $\tilde{\varphi}$ has continuous derivatives up to the second order in the considered part of $\tilde{\Omega}$.

Proof. Let us first realize that for (ξ, η) on the boundary of $\tilde{\Omega}$ the Lebesgue measure of the set $Q(\xi, \eta) \cap \tilde{\Omega}$ is zero so the first initial condition (3) and boundary conditions (4) are satisfied if equation (38) holds.

We use equations (23)–(25) for several ranges of variables ξ, η. We denote as G_0 the function (24) and (25) corresponding to the d'Alembert equation solution $\tilde{\varphi}^{(0)}$ with zero mass. As for $T(B(\xi, \eta)) \leqslant 0$

$$F^{-1}(\xi) \leqslant -\eta \leqslant a(0) \tag{39}$$

(see equations (8) and (30)), $\xi \leqslant F(a(0))$. From equations (9) and (39) we see that $F^{-1}(\xi) \leqslant \eta \leqslant -F^{-1}(\xi)$, i.e.

$$F^{-1}(\xi) \leqslant 0 \qquad |\eta| \leqslant |F^{-1}(\xi)|. \tag{40}$$

Furthermore, it is clear that ξ and η cannot both be simultaneously greater than $a(0)$ as $T(B(\xi, \eta))$ would be positive in this case (remember that $F^{-1}(a(0)) = -a(0)$). Taking into account equation (9) we see that always $\eta \leqslant a(0)$ under the assumptions of the lemma. We have to distinguish two cases now.

(1) $\xi \leqslant a(0), \eta \leqslant a(0)$

Now equation (24) can be used in equation (23) and we obtain

$$\begin{aligned}\tilde{\varphi}(\xi,\eta) &= \tilde{\varphi}^{(0)}(\xi,\eta) + \frac{m^2}{4}\left[\int_{|\eta|}^{\xi} dy \int_{\eta}^{y} dz + \int_{0}^{|\eta|} dy \int_{-y}^{y} dz - \int_{0}^{\xi} dy \int_{-y}^{y} dz\right]\tilde{\varphi}(y,z) \\ &= \tilde{\varphi}^{(0)}(\xi,\eta) + \frac{m^2}{4}\left[\int_{|\eta|}^{\xi} dy \int_{\eta}^{y} dz - \int_{|\eta|}^{\xi} dy \int_{-y}^{y} dz\right]\tilde{\varphi}(y,z) \\ &= \tilde{\varphi}^{(0)}(\xi,\eta) - \frac{m^2}{4}\int_{|\eta|}^{\xi} dy \int_{-y}^{\eta} dz\, \tilde{\varphi}(y,z)\end{aligned}$$

which is equation (38) due to equation (35) as here $F^{-1}(\xi) \leqslant -a(0) \leqslant -\xi \leqslant -y$. On the other hand, differentiating the last equation and using the already proven boundary conditions, we can verify that $\tilde{\varphi}$ has continuous second derivatives and satisfies equation (13) if the integral relation (38) holds. The initial condition for the time derivative in equation (3) is also seen.

(2) $a(0) < \xi \leqslant F(a(0)), \eta \leqslant a(0)$

Now formula (24) holds for $G(\eta)$ while equation (25) gives

$$\begin{aligned}G(\xi) &= G(F^{-1}(\xi)) + \frac{m^2}{4}\int_{|F^{-1}(\xi)|}^{\xi} dy \int_{F^{-1}(\xi)}^{y} dz\, \tilde{\varphi}(y,z) \\ &= G_0(\xi) + \frac{m^2}{4}\left[\int_{0}^{|F^{-1}(\xi)|} dy \int_{-y}^{y} dz + \int_{|F^{-1}(\xi)|}^{\xi} dy \int_{F^{-1}(\xi)}^{y} dz\right]\tilde{\varphi}(y,z)\end{aligned}$$

and

$$\begin{aligned}\tilde{\varphi}(\xi,\eta) = \tilde{\varphi}^{(0)}(\xi,\eta) + \frac{m^2}{4}&\left[\int_{|\eta|}^{\xi} dy \int_{\eta}^{y} dz + \int_{0}^{|\eta|} dy \int_{-y}^{y} dz\right. \\ &\left. - \int_{0}^{|F^{-1}(\xi)|} dy \int_{-y}^{y} dz - \int_{|F^{-1}(\xi)|}^{\xi} dy \int_{F^{-1}(\xi)}^{y} dz\right]\tilde{\varphi}(y,z).\end{aligned}$$

With the help of equation (40) we combine the second and third integrals and the first and fourth integrals separating the first into two parts. Then

$$\begin{aligned}\tilde{\varphi}(\xi,\eta) &= \tilde{\varphi}^{(0)}(\xi,\eta) + \frac{m^2}{4}\left[\int_{|\eta|}^{|F^{-1}(\xi)|} dy \int_{\eta}^{y} dz\right. \\ &\qquad \left. - \int_{|F^{-1}(\xi)|}^{\xi} dy \int_{F^{-1}(\xi)}^{\eta} dz - \int_{|\eta|}^{|F^{-1}(\xi)|} dy \int_{-y}^{y} dz\right]\tilde{\varphi}(y,z) \\ &= \tilde{\varphi}^{(0)}(\xi,\eta) + \frac{m^2}{4}\left[-\int_{|\eta|}^{|F^{-1}(\xi)|} dy \int_{-y}^{\eta} dz - \int_{|F^{-1}(\xi)|}^{\xi} dy \int_{F^{-1}(\xi)}^{\eta} dz\right]\tilde{\varphi}(y,z) \\ &= \tilde{\varphi}^{(0)}(\xi,\eta) - \frac{m^2}{4}\int_{|\eta|}^{\xi} dy \int_{\max(F^{-1}(\xi),-y)}^{\eta} dz\, \tilde{\varphi}(y,z)\end{aligned}$$

which is equation (38). On the other hand, differentiating the next-to-last equation and using the already proven boundary conditions, we can verify that $\tilde{\varphi}$ has continuous second derivatives and satisfies equation (13) if the integral relation (38) holds.

The continuity of the first and second derivatives at $\xi = a(0)$ can be also seen. □

Let us denote the iterations of the map B as

$$B^0(\xi,\eta) = (\xi,\eta),\ B^1(\xi,\eta) = B(\xi,\eta),\ B^2(\xi,\eta) = B(B(\xi,\eta)),\ldots$$

and let

$$N(\xi,\eta) = \max\{n \in \mathbb{Z} | B^n(\xi,\eta) \in \tilde{\Omega}\} \tag{41}$$

$$M(\xi,\eta) = \bigcup_{n=0}^{N(\xi,\eta)-1} Q(B^n(\xi,\eta)) \cup (Q(B^{N(\xi,\eta)}(\xi,\eta)) \cap \tilde{\Omega}) \tag{42}$$

(see figure 1)

$$\vartheta(\xi,\eta,y,z) = (-1)^{n+1} \tag{43}$$

for $(y,z) \in Q(B^n(\xi,\eta))$, $n \in \mathbb{Z}$.

Theorem 1. *Let $\tilde{\varphi}^{(0)}$ satisfy equation (13) with m replaced by zero and boundary conditions corresponding to equations (3) and (4). If $\tilde{\varphi} \in C^2(\tilde{\Omega})$ satisfies equation (13) with boundary conditions corresponding to equations (3) and (4) then*

$$\tilde{\varphi}(\xi,\eta) = \tilde{\varphi}^{(0)}(\xi,\eta) + \frac{m^2}{4}\int_{M(\xi,\eta)} \vartheta(\xi,\eta,y,z)\tilde{\varphi}(y,z)\,\mathrm{d}y\,\mathrm{d}z. \tag{44}$$

Conversely, if $\tilde{\varphi} \in C^0(\tilde{\Omega})$ satisfies equation (44) then $\tilde{\varphi} \in C^2(\tilde{\Omega})$, $\tilde{\varphi}$ satisfies equation (13) and the boundary conditions corresponding to equations (3) and (4).

Proof. Let $(\xi,\eta) \in \tilde{\Omega}$. Using equation (36), $N(\xi,\eta)$ defined by equation (41) exists since $\inf\{a(t) | 0 \leqslant t \leqslant T(\xi,\eta)\} > 0$. With the help of lemmas 2 and 3, the theorem follows by induction with respect to $N(\xi,\eta)$. The continuity of $\tilde{\varphi}$ and its derivatives up to the second order at the curve $T(B(\xi,\eta)) = 0$ separating the regions of validity of the two lemmas can be also checked. □

Lemma 4. *Let $a_{\max} := \sup a < \infty$ and $(\xi,\eta) \in \tilde{\Omega}$. Then*

$$M(\xi,\eta) \subset \{(y,z) \in \mathbb{R}^2 | 0 \leqslant y \leqslant \xi,\ y - 2a_{\max} \leqslant z \leqslant y\} \tag{45}$$

and the Lebesgue measure of the set $M(\xi,\eta)$ verifies

$$0 \leqslant \int_{M(\xi,\eta)} \mathrm{d}y\,\mathrm{d}z \leqslant 2a_{\max}T(\xi,\eta) \leqslant 2a_{\max}\xi. \tag{46}$$

If (ξ,η) is on the boundary of $\tilde{\Omega}$ then the Lebesgue measure of $M(\xi,\eta)$ is zero.

Proof. Let $(y,z) \in M(\xi,\eta)$. Using equation (31), $y \leqslant \xi$ for $(y,z) \in Q(\xi,\eta)$. Looking at equations (9), (32) and (42), the inequality $y \leqslant \xi$ is seen for all $(y,z) \in M(\xi,\eta)$. Furthermore, $y \geqslant 0$ as $M(\xi,\eta) \subset \tilde{\Omega}$ (see again equation (9)). Using equation (31), $z \leqslant \eta \leqslant y$ and

$$z \geqslant F^{-1}(\xi) \geqslant F^{-1}(y) = y - 2a \circ k^{-1}(y) \geqslant y - 2a_{\max}$$

for arbitrary $(y,z) \in Q(\xi,\eta)$, $(\xi,\eta) \in \tilde{\Omega}$. The estimate (46) now immediately follows using variables t and x to calculate the first bound.

The boundary of $\tilde{\Omega}$ consists of points for which:

(i) $\xi = -\eta$, i.e. $t = 0$;
(ii) $\xi = \eta$, i.e. $x = 0$;
(iii) $\xi = F(\eta)$, i.e. $x = a(t)$.

In case (i), $M(\xi, \eta)$ is one-point and therefore the measure is zero. In cases (ii) and (iii), the measure of $Q(\xi, \eta)$ is zero by equation (31) and $B(\xi, \eta)$ is also on the boundary of $\tilde{\Omega}$ if still $B(\xi, \eta) \in \tilde{\Omega}$. So the measure of $M(\xi, \eta)$ is zero according to equation (42). □

Theorem 2. *Let* $a \in \mathrm{C}^2([0, \infty))$, $0 < \inf a \leqslant \sup a < \infty$, $|a'| < 1$, $\varphi_0 \in \mathrm{C}^2([0, a(0)])$, $\varphi_1 \in \mathrm{C}^1([0, a(0)])$ *and the following relations are satisfied:*

$$\varphi_0(0) = 0 \qquad \varphi_0(a(0)) = 0 \tag{47}$$

$$\varphi_1(0) = 0 \qquad \varphi_1(a(0)) + a'(0)\varphi_0'(a(0)) = 0 \tag{48}$$

$$\varphi_0''(0) = 0 \tag{49}$$

$$(1 + a'(0)^2)\varphi_0''(a(0)) + a''(0)\varphi_0'(a(0)) + 2a'(0)\varphi_1'(a(0)) = 0. \tag{50}$$

Then there exists unique $\tilde{\varphi} \in \mathrm{C}^2(\tilde{\Omega})$ *satisfying equation (13) in* $\tilde{\Omega}$ *and boundary conditions corresponding to the transformed relations (3) and (4). The solution satisfies the estimate*

$$|\tilde{\varphi}(\xi, \eta)| \leqslant c\, \mathrm{e}^{\frac{1}{2}a_{\max} m^2 \xi} \tag{51}$$

with a constant $0 < c < \infty$ *independent of m (but dependent on* φ_0 *and* φ_1*).*

Proof. We iterate equation (44) denoting

$$\tilde{\varphi}^{(n)}(\xi, \eta) = \tilde{\varphi}^{(0)}(\xi, \eta) + \frac{m^2}{4} \int_{M(\xi,\eta)} \vartheta(\xi, \eta, y, z)\tilde{\varphi}^{(n-1)}(y, z)\, \mathrm{d}y\, \mathrm{d}z \tag{52}$$

$$\varepsilon^{(n-1)}(\xi, \eta) = \tilde{\varphi}^{(n)}(\xi, \eta) - \tilde{\varphi}^{(n-1)}(\xi, \eta) \tag{53}$$

for $n = 1, 2, 3, \ldots$. Now

$$\varepsilon^{(n)}(\xi, \eta) = \frac{m^2}{4} \int_{M(\xi,\eta)} \vartheta(\xi, \eta, y, z)\varepsilon^{(n-1)}(y, z)\, \mathrm{d}y\, \mathrm{d}z. \tag{54}$$

Under our assumptions, all the functions $\tilde{\varphi}^{(n)}$ and $\varepsilon^{(n)}$ are continuous. Then the estimate

$$|\varepsilon^{(n)}(\xi, \eta)| \leqslant \frac{a_{\max}^n m^{2n} \xi^n}{2^n n!} \|\varepsilon^{(0)}\|_{L^\infty(M_0)} \tag{55}$$

follows by induction using lemmas 1 and 4 for

$$(\xi, \eta) \in M_0 := \{(y, z) \in \tilde{\Omega} | T(y, z) \leqslant T_0\}$$

with arbitrary given $T_0 > 0$. So the sequence

$$\tilde{\varphi}^{(n+1)} = \sum_{k=0}^{n} \varepsilon^{(k)} + \tilde{\varphi}^{(0)} \tag{56}$$

is uniformly convergent in any compact subset of $\tilde{\Omega}$. Its limit $\tilde{\varphi}$ is continuous, satisfying equation (44) and the required boundary and initial conditions. Therefore, $\tilde{\varphi}$ also satisfies equation (13).

Let us assume that we have two solutions of equation (13) satisfying the required boundary and initial conditions. Then they satisfy also equation (44) and the estimate such as (55) holds for their difference with any n. So the two solutions must be identical and the uniqueness is proven.

The solution $\tilde{\varphi}^{(0)}$ of the d'Alembert equation is known to be bounded in $\tilde{\Omega}$ (see also equations (23) and (25) for $f = 0$). Equation (52) then leads to

$$|\tilde{\varphi}(\xi, \eta)| \leqslant \|\tilde{\varphi}^{(0)}\|_\infty\, \mathrm{e}^{\frac{1}{2}a_{\max} m^2 \xi} \tag{57}$$

and equation (51) is proven. □

5. Energy large-time behaviour

In this section, we prove that the energy exponentially increase (up to non-monotone evolution within the period of the end-point motion) for sufficiently small mass under the same assumptions as for the massless case. Let us first write a formula for the function G by iterations of relation (25). Let G_0 be the corresponding function for $m = 0$. For any $\eta \in [-a(0), \infty)$ there exists just one non-negative integer $n(\eta)$ such that

$$\eta \in F^{n(\eta)}([-a(0), a(0))). \tag{58}$$

We also use integer

$$K(t) = n(t + a_{\max}) \geqslant n(\xi) \geqslant n(\eta). \tag{59}$$

By induction with respect to $n(\eta)$ and comparison of the relations with $m = 0$ and $m \neq 0$ we obtain

$$G(\eta) = G_0(\eta) + \frac{m^2}{4}\int_0^{|F^{-n(\eta)}(\eta)|} \mathrm{d}y \int_{-y}^{y} \mathrm{d}z\, \tilde{\varphi}(y, z) + \frac{m^2}{4}\sum_{j=1}^{n(\eta)} \int_{|F^{-j}(\eta)|}^{F^{1-j}(\eta)} \mathrm{d}y \int_{F^{-j}(\eta)}^{y} \mathrm{d}z\, \tilde{\varphi}(y, z). \tag{60}$$

To calculate the energy density, we also need derivatives of the function $\tilde{\varphi}$ and therefore G. Taking into account that $F^{-j}(\eta)$ can be negative only for $j = n(\eta)$ and excluding a discrete set of values of η, we obtain

$$\begin{aligned} G'(\eta) = G_0'(\eta) + \frac{m^2}{4}\Bigg\{ & \frac{\mathrm{d}F^{-n(\eta)}(\eta)}{\mathrm{d}\eta}\operatorname{sgn}(F^{-n(\eta)}(\eta)) \\ & \times \int_{-|F^{-n(\eta)}(\eta)|}^{\delta_{n(\eta),0}|F^{-n(\eta)}(\eta)|+(1-\delta_{n(\eta),0})F^{-n(\eta)}(\eta)} \tilde{\varphi}(|F^{-n(\eta)}(\eta)|, z)\,\mathrm{d}z \\ & + \sum_{j=1}^{n(\eta)} \frac{\mathrm{d}F^{1-j}(\eta)}{\mathrm{d}\eta} \int_{F^{-j}(\eta)}^{F^{1-j}(\eta)} \tilde{\varphi}(F^{1-j}(\eta), z)\,\mathrm{d}z \\ & - \sum_{j=1}^{n(\eta)} \frac{\mathrm{d}F^{-j}(\eta)}{\mathrm{d}\eta} \int_{|F^{-j}(\eta)|}^{F^{1-j}(\eta)} \tilde{\varphi}(y, F^{-j}(\eta))\,\mathrm{d}y \Bigg\}. \end{aligned} \tag{61}$$

This formula has the form

$$G'(\eta) = G_0'(\eta) + \frac{m^2}{4}\sum_{j=0}^{n(\eta)} B_j(\eta)\frac{\mathrm{d}F^{-j}(\eta)}{\mathrm{d}\eta} \tag{62}$$

where $B_j(\eta)$ are seen above. Using equation (51) and the relation

$$\eta - F^{-1}(\eta) = 2a \circ k^{-1}(\eta) \leqslant 2a_{\max} \tag{63}$$

following from equation (6), the estimate

$$|B_j(\eta)| \leqslant c_1(m)\,\mathrm{e}^{\frac{1}{2}a_{\max}m^2|F^{-j}(\eta)|} \tag{64}$$

is shown for $j = 0, \ldots, n(\eta) > 0$ with

$$c_1(m) = 2a_{\max}c\left(1 + \mathrm{e}^{a_{\max}^2 m^2}\right). \tag{65}$$

Here, we indicate the dependence of constant c_1 on the mass m but we do not indicate the automatically assumed dependence on the initial data φ_0, φ_1 (see equations (51) and (57)) and the function a. We keep such a notation for further constants in estimates below, as we finally want to have an m-independent estimate over the range of mass values. Using equations (58) and (63),

$$|F^{-j}(\eta)| \leqslant [2n(\eta) - 2j + 1]a_{\max} \tag{66}$$

for $j = 0, \ldots, n(\eta)$ and therefore

$$|B_j(\eta)| \leqslant c_1(m)\, \mathrm{e}^{(n(\eta)-j+\frac{1}{2})a_{\max}^2 m^2}. \tag{67}$$

Using the above formulae together with equation (23) we obtain for $\omega = \xi, \eta$

$$\frac{\partial\tilde{\varphi}(\xi,\eta)}{\partial\omega} = \frac{\partial\tilde{\varphi}^{(0)}(\xi,\eta)}{\partial\omega} + \frac{m^2}{4}\sum_{j=0}^{n(\omega)} A_j^{(\omega)}(\xi,\eta)\frac{\mathrm{d}F^{-j}(\omega)}{\mathrm{d}\omega} \tag{68}$$

where

$$A_0^{(\xi)}(\xi,\eta) = \int_\eta^\xi \tilde{\varphi}(\xi,z)\,\mathrm{d}z - B_0(\xi) \qquad A_j^{(\xi)}(\xi,\eta) = -B_j(\xi) \tag{69}$$

$$A_0^{(\eta)}(\xi,\eta) = -\mathrm{sgn}(\eta)\int_\eta^{|\eta|} \tilde{\varphi}(|\eta|,z)\,\mathrm{d}z - \int_{|\eta|}^\xi \tilde{\varphi}(y,\eta)\,\mathrm{d}y + B_0(\eta) \tag{70}$$

$$A_j^{(\eta)}(\xi,\eta) = B_j(\eta) \tag{71}$$

for $j = 1, \ldots, n(\omega)$. The upper estimate of $2ca_{\max}\,\mathrm{e}^{\frac{1}{2}a_{\max}m^2\xi}$ for integral terms in both formulae for $A_0^{(\omega)}(\xi,\eta)$ can be seen from equation (51). Realizing that $n(\eta) \leqslant n(\xi)$ as $\eta \leqslant \xi$ and that $\xi < (2n(\xi)+1)a_{\max}$ according to equation (66), we arrive at

$$\left|A_j^{(\omega)}(\xi,\eta)\right| \leqslant c_2(m)\,\mathrm{e}^{(n(\xi)-j)a_{\max}^2 m^2} \tag{72}$$

$$c_2(m) = 2a_{\max}c\left(2 + \mathrm{e}^{a_{\max}^2 m^2}\right)\mathrm{e}^{\frac{1}{2}a_{\max}^2 m^2} \tag{73}$$

for $j = 0, \ldots, n(\omega)$ and $\omega = \xi, \eta$.

To estimate the contributions of terms in equation (68) to the energy (14), we use the results for the d'Alembert equation [1, 5]. Let us first recall these in a suitable form. Function F, defined in equation (5), is an increasing diffeomorphism of the line $\mathbb{R}$ satisfying the relation $F(t+T) = F(t) + T$ for $t \in R$, i.e. a covering of a diffeomorphism of the circle of length T. The notions of the rotation number and periodic point used below are defined, for example, in [10]; a brief review is also given in [5]. We use the notation

$$F^n = F \circ \cdots \circ F$$

for the composition of the function F n-times with itself, $(f)^n$ for the nth power of the function f, i.e. for the function with values

$$f(x)^n = f(x) \cdot \ldots \cdot f(x)$$

$DF = F'$ for the derivative of function F, and χ_I for the characteristic function of the interval I with the value equal to one in I and zero outside I.

Lemma 5. *Function F has the rotation number $\rho(F) = \frac{p}{q}T$ where $p \in \mathbb{N}^* = \{1, 2, \ldots\}$ and $q \in \mathbb{N}^*$ are relatively prime; a_1 is an attracting periodic point of F; the function F has in $[a_1, F(a_1))$ a finite number of periodic points of period q from which $a_1 < a_2 < \cdots < a_N$ are attracting with*

$$DF^q(a_i) < 1 \qquad \text{for} \quad i = 1, \ldots, N$$

and other periodic points in $[a_1, F(a_1))$ *are repelling. Let us denote as* b_{i-1}, b_i *the nearest repelling periodic points to* a_i *such that* $b_{i-1} < a_i < b_i$*, and as* $J_1 = [a_1, b_1) \cup (b_N, F(a_1))$*,* $J_i = (b_{i-1}, b_i)$ *for* $i = 2, \ldots, N$*. Let* $f \in L^2((a_1, F(a_1)))$ *be a real function,* $\|f\| > 0$*. Then*

$$\int_{a_1}^{F(a_1)} \frac{f(x)^2}{DF^{nq}(x)}\,\mathrm{d}x = \sum_{i=1}^{N} A_i [DF^q(a_i)]^{-n} + R_n \tag{74}$$

where $0 \leqslant A_i < \infty$*,* $A_i > 0$ *if* $\|f\|_{L^2(J_i)} > 0$*,*

$$A_i = \|\sqrt{l^{(i)}} f\|^2_{L^2(J_i)} \qquad l^{(i)}(x) = \prod_{k=0}^{\infty} \frac{DF^q(a_i)}{DF^q \circ F^{kq}(x)} \tag{75}$$

for $x \in J_i$ *and* $i = 1, \ldots, N$*. The remainder*

$$R_n = o\big(\big[DF^q\big(a_{i_0}\big)\big]^{-n}\big) \tag{76}$$

as $n \to \infty$*,* i_0 *being defined by the relation*

$$DF^q(a_{i_0}) = \min\{DF^q(a_i) | i = 1, \ldots, N \text{ and } A_i > 0\}. \tag{77}$$

Proof. The set of periodic points b satisfying

$$F^q(b) = b + pT$$

the set of attracting periodic points, and the set of repelling periodic points are invariant under the action of function F. Furthermore

$$DF^q(b) = DF^q(F^n(b))$$

for any periodic point b and $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Under our assumptions, necessarily

$$a_1 < b_1 < a_2 < \cdots < b_{N-1} < a_N < b_N < F(a_1)$$

are just all the periodic points in $[a_1, F(a_1)]$. Writing

$$\int_{a_1}^{F(a_1)} \frac{(f)^2}{DF^{nq}}\,\mathrm{d}x = \sum_{i=1}^{N} \int_{J_i} \frac{(f)^2}{DF^{nq}}\,\mathrm{d}x \tag{78}$$

all the terms can be treated as in the proof of theorem 3.25 of [5] applied to the function F^q. We know from [5] that the sequence of functions

$$l_n^{(i)} = \prod_{j=0}^{n-1} \frac{DF^q(a_i)}{DF^q \circ F^{jq}}$$

is uniformly bounded and pointwise convergent in J_i to the strictly positive function $l^{(i)}$ as $n \to \infty$. So the limit of each term in equation (78) multiplied by $[DF^q(a_i)]^n$ can be calculated taking the limit under the integral and the formulae (74)–(76) are obtained. As $\|f\|_{L^2(a_1, F(a_1))} > 0$, an index i_0 surely exists and the leading term is nontrivial. □

Remark. The validity of lemma 5 was mentioned in [5] but only the case of integer rotation number ($\rho(F) = pT$ where $p \in \mathbb{N}^*$) and two periodic points in $[-a(0), a(0))$ was explicitly written for simplicity. However, the assumption of only two periodic points then leads to $p = 1$ as can be seen using the invariance of the set of periodic points under F and under the translation by period T. We have overlooked this constraint in [5].

Let us now repeat some assumptions and formulate some new assumptions for the purpose of the main theorem.

Assumptions. Let $a \in C^2(\mathbb{R})$ be a strictly positive periodic function with a period $T > 0$, satisfying $|a'| < 1$. The function F defined by relations (5) has the rotation number

$$\rho(F) = \frac{p}{q} \tag{79}$$

where $p, q \in \mathbb{N}^* = \{1, 2, \ldots\}$ are relatively prime. The function F has a finite number of periodic points in the interval $I_0 = [-a(0), a(0))$, of them $a_1, \ldots, a_N$ attracting ($N \in \mathbb{N}^*$) and other periodic points repelling. Let us denote as $b_0, \ldots, b_N$ the repelling periodic points such that

$$b_0 < a_1 < b_1 < \cdots < b_{N-1} < a_N < b_N = F(b_0) \tag{80}$$

and there are no other periodic points in (b_0, b_N). Notice that these repelling periodic points necessarily exist, equation (80) is the only possible ordering of periodic points, and just one of the two periodic points b_0, b_N lies in I_0. As the initial data determine the function G directly in the interval I_0 by equation (24), while the whole intervals (b_0, b_1) and (b_{N-1}, b_N) need not be a part of I_0, we have to map the outreaching part into I_0 by functions F and F^{-1} respectively if necessary. Let us therefore denote

$$\begin{aligned} J_1 &= (\max\{b_0, -a(0)\}, b_1) \cup (\min\{b_N, a(0)\}, a(0)) \\ J_i &= (b_{i-1}, b_i) \qquad \text{for} \quad i = 2, \ldots, N-1 \\ J_N &= (b_{N-1}, \min\{b_N, a(0)\}) \cup (-a(0), \max\{b_0, -a(0)\}) \end{aligned} \tag{81}$$

where notation $(x, y) = \emptyset$ for $x \geqslant y$ is used. These formulae can be also written in a more compact and, for further use, clearer way as

$$J_i = \mathring{I}_0 \bigcap \bigcup_{j=-1}^{1} F^j((b_{i-1}, b_i)) \qquad \text{for} \quad i = 1, \ldots, N.$$

Let all the attracting periodic points be such that

$$DF^q(a_i) < 1 \qquad \text{for} \quad i = 1, \ldots, N \tag{82}$$

and let index i_0 be such that

$$DF^q(a_{i_0}) = \min\{DF^q(a_i) | i = 1, \ldots, N\}. \tag{83}$$

Let us denote

$$J = \cup\big\{J_i | DF^q(a_i) = DF^q\big(a_{i_0}\big)\big\}. \tag{84}$$

We are ready to formulate the main statement now.

Theorem 3. *Let functions a, φ_0 and φ_1 be as in theorem 2 and satisfy the assumptions given above. The function $\varphi_0'(|x|) + \varphi_1(|x|)\,\mathrm{sgn}(x)$ has a strictly positive norm in $L^2(J)$. Then there exist strictly positive finite constants m_0, A and B (dependent on φ_0 and φ_1) such that for masses $0 \leqslant m \leqslant m_0$ and time $t \geqslant 0$ the energy $E_m(t)$ satisfies the inequality*

$$A\,\mathrm{e}^{\gamma t} \leqslant E_m(t) \leqslant B\,\mathrm{e}^{\gamma t} \tag{85}$$

where

$$\gamma = -\frac{1}{pT} \ln\big(DF^q\big(a_{i_0}\big)\big) > 0. \tag{86}$$

Proof. Firstly, let us show that the terms defining the energy do not change by more than a constant factor if time undergoes a constant translation. This enables us to use a special sequence of times only. Let us consider two times t_1 and t_2 satisfying the inequalities

$$h(t_1) \leqslant h(t_2) \leqslant k(t_1) \tag{87}$$

i.e. $t_1 \leqslant t_2 \leqslant h^{-1} \circ k(t_1)$. The last inequality is clearly satisfied if

$$t_1 \leqslant t_2 \leqslant t_1 + 2a_{\min} \tag{88}$$

as the second inequality (87) reads $t_2 \leqslant t_1 + a(t_1) + a(t_2)$. Similar calculations as in the proof of lemma 2.17 of [5] lead to the estimate for the energy of the massless field

$$\frac{1}{F'_{\max}} E_0(t_1) \leqslant E_0(t_2) \leqslant \frac{1}{F'_{\min}} E_0(t_1). \tag{89}$$

Analogously for

$$S_j(t) := \int_{h(t)}^{k(t)} (DF^{-j}(y))^2 \, \mathrm{d}y \tag{90}$$

where $j \in \mathbb{N}$, we can write

$$\begin{aligned} S_j(t_1) &= \left(\int_{h(t_1)}^{h(t_2)} + \int_{h(t_2)}^{k(t_1)} \right) DF^{-j}(y)^2 \, \mathrm{d}y \\ &= \int_{h(t_2)}^{k(t_1)} DF^{-j}(y)^2 \, \mathrm{d}y + \int_{k(t_1)}^{k(t_2)} (DF^{-j} \circ F^{-1}(y))^2 DF^{-1}(y) \, \mathrm{d}y \\ &= \int_{h(t_2)}^{k(t_1)} DF^{-j}(y)^2 \, \mathrm{d}y + \int_{k(t_1)}^{k(t_2)} DF^{-j}(y)^2 \frac{(DF^{-1} \circ F^{-j}(y))^2}{DF^{-1}(y)} \, \mathrm{d}y. \end{aligned}$$

Estimating the fraction in the last integrand in terms of $F'_{\min} < 1 < F'_{\max}$ and the factor 1 in the first integral by the same value, we obtain

$$\frac{F'^2_{\min}}{F'_{\max}} S_j(t_1) \leqslant S_j(t_2) \leqslant \frac{F'^2_{\max}}{F'_{\min}} S_j(t_1). \tag{91}$$

Let us denote

$$t_0 = h^{-1}(a_{i_0}). \tag{92}$$

Combining inequalities (88), (89) and (91) we see that for

$$npT + t_0 \leqslant t \leqslant (n+1)pT + t_0 \tag{93}$$

with any $n \in \mathbb{N}$, the estimates

$$L_1 E_0(npT + t_0) \leqslant E_0(t) \leqslant L_2 E_0(npT + t_0) \tag{94}$$

$$M_1 S_j(npT + t_0) \leqslant S_j(t) \leqslant M_2 S_j(npT + t_0) \tag{95}$$

hold where

$$\begin{aligned} &L_1 = F'^{-n_0}_{\max} \qquad L_2 = F'^{-n_0}_{\min} \qquad M_1 = \left(\frac{F'^2_{\min}}{F'_{\max}} \right)^{n_0} \\ &M_2 = \left(\frac{F'^2_{\max}}{F'_{\min}} \right)^{n_0} \qquad n_0 = \left[\frac{pT}{2a_{\min}} \right] + 1. \end{aligned} \tag{96}$$

Here, the square brackets denote the entire part.

We denote

$$\tilde{E}(t) = \sum_{\omega=\xi,\eta} \left\| \frac{\partial \tilde{\varphi}}{\partial \omega} \right\|^2_{L^2((0,a(t)),\mathrm{d}x)} \tag{97}$$

and we write equation (68) as

$$\frac{\partial \tilde{\varphi}}{\partial \omega} = \frac{\partial \tilde{\varphi}^{(0)}}{\partial \omega} + \kappa_\omega. \tag{98}$$

Now

$$\sqrt{E_0(t)} - \sqrt{\sum_{\omega=\xi,\eta} \|\kappa_\omega\|^2} \leqslant \sqrt{\tilde{E}(t)} \leqslant \sqrt{E_0(t)} + \sqrt{\sum_{\omega=\xi,\eta} \|\kappa_\omega\|^2} \tag{99}$$

by the triangle inequality. Using estimates (72),

$$\begin{aligned}\sqrt{\sum_{\omega=\xi,\eta} \|\kappa_\omega\|^2} &\leqslant \frac{m^2}{4} c_2(m) \sum_{j=0}^{K(t)} \mathrm{e}^{(K(t)-j)a_{\max}^2 m^2} \|DF^{-j}\|_{L^2(h(t),k(t))} \\ &\leqslant \frac{m^2}{4} c_2(m) \sqrt{M_2} \sum_{j=0}^{K(t)} \mathrm{e}^{(K(t)-j)a_{\max}^2 m^2} \sqrt{S_j(npT+t_0)}\end{aligned} \tag{100}$$

for t satisfying equation (93). For $j = iq + r$ with $i \in \mathbb{N}$ and $r = 0, \ldots, q-1$ we calculate

$$\begin{aligned} S_j(npT+t_0) &= \int_{npT+a_{i_0}}^{npT+F(a_{i_0})} DF^{-j}(y)^2 \,\mathrm{d}y = \int_{a_{i_0}}^{F(a_{i_0})} DF^{-j}(y)^2 \,\mathrm{d}y \\ &= \int_{F^{-j}(a_{i_0})}^{F^{1-j}(a_{i_0})} DF^{-j} \circ F^j(y) \,\mathrm{d}y = \int_{F^{-r}(a_{i_0})}^{F^{1-r}(a_{i_0})} \frac{\mathrm{d}y}{DF^j(y)} \\ &= \int_{a_{i_0}}^{F(a_{i_0})} \frac{DF^{-r}(y)^2}{DF^{j-r}(y)} \,\mathrm{d}y \leqslant F_{\min}^{\prime -2r} \int_{a_{i_0}}^{F(a_{i_0})} \frac{\mathrm{d}y}{DF^{iq}(y)}. \end{aligned} \tag{101}$$

Using lemma 5 there exists a finite constant $c_3 > 0$ such that

$$0 < \int_{a_{i_0}}^{F(a_{i_0})} \frac{\mathrm{d}y}{DF^{iq}(y)} < c_3 \left[DF^q\left(a_{i_0}\right)\right]^{-i} \leqslant c_3 \left[DF^q\left(a_{i_0}\right)\right]^{-\frac{j}{q}}. \tag{102}$$

Equation (100) now gives

$$\sqrt{\sum_{\omega=\xi,\eta} \|\kappa_\omega\|^2} \leqslant \frac{m^2}{4} c_2(m) \sqrt{c_3 M_2} F_{\min}^{\prime -(q-1)} \,\mathrm{e}^{K(t) a_{\max}^2 m^2} \frac{\left(DF^q\left(a_{i_0}\right)^{-\frac{1}{2q}} \mathrm{e}^{-a_{\max}^2 m^2}\right)^{K(t)+1} - 1}{DF^q\left(a_{i_0}\right)^{-\frac{1}{2q}} \mathrm{e}^{-a_{\max}^2 m^2} - 1}. \tag{103}$$

We now choose $m_1 > 0$ such that

$$m_1^2 < -\frac{1}{2q a_{\max}^2} \ln\left(DF^q\left(a_{i_0}\right)\right) \tag{104}$$

and we consider only mass values

$$0 \leqslant m \leqslant m_1. \tag{105}$$

Now we can estimate

$$\sqrt{\sum_{\omega=\xi,\eta} \|\kappa_\omega\|^2} \leqslant c_4 m^2 \left(DF^q\left(a_{i_0}\right)\right)^{-\frac{K(t)+1}{2q}} \tag{106}$$

where we denote

$$c_4 = \frac{\frac{1}{4} c_2(m_1) \sqrt{c_3 M_2} F_{\min}^{\prime -(q-1)}}{DF^q\left(a_{i_0}\right)^{-\frac{1}{2q}} \mathrm{e}^{-a_{\max}^2 m_1^2} - 1} \,\mathrm{e}^{-a_{\max}^2 m_1^2}. \tag{107}$$

The energy of the massless field is given by function G_0 defined by equations (24) and (25) with $f = 0$. For the considered sequence of times, it reads

$$E_0(npT + t_0) = \int_{npT+a_{i_0}}^{npT+F(a_{i_0})} G_0'(y)^2\,dy$$
$$= \int_{F^{nq}(a_{i_0})}^{F^{nq+1}(a_{i_0})} G_0'(y)^2\,dy = \int_{a_{i_0}}^{F(a_{i_0})} \frac{G_0'(y)^2}{DF^{nq}(y)}\,dy. \quad (108)$$

Now lemma 5 and inequality (94) show the existence of constants $0 < D_1 < D_2 < \infty$ such that

$$D_1\big(DF^q(a_{i_0})\big)^{-n} \leqslant E_0(t) \leqslant D_2\big(DF^q(a_{i_0})\big)^{-n} \quad (109)$$

for t satisfying equation (93) as it is clear from the prolongation formula for G_0 (25) with $f = 0$, that the assumed nontriviality of $\varphi_0(|x|) + \varphi_1'(|x|)\,\mathrm{sgn}(x)$ in J leads to the nontriviality of G_0' in a suitable neighbourhood of a suitable attracting periodic point of F in $[a_{i_0}, F(a_{i_0}))$.

Let us relate the numbers n in equation (93) and $K(t)$ defined by equations (58) and (59). As equation (93) gives

$$t + a_{\max} \geqslant k(t) = F(h(t)) \geqslant F(npT + a_{i_0}) \geqslant F^{nq+1}(-a(0))$$

we have

$$K(t) \geqslant nq + 1. \quad (110)$$

Similarly equation (93) implies

$$t + a_{\max} = h(t) + a(t) + a_{\max} \leqslant h(t) + 2a_{\max} \leqslant a_{i_0} + (n+1)pT$$
$$+ 2a_{\max} < a_{i_0} + \left(n + \left[\frac{2a_{\max}}{pT}\right] + 2\right) pT < F^{(n+[\frac{2a_{\max}}{pT}]+2)q}(a(0))$$

where square brackets denote the entire part and

$$K(t) \leqslant \left(n + \left[\frac{2a_{\max}}{pT}\right] + 2\right) q. \quad (111)$$

The estimate of equation (109) can be written as

$$D_1'\big[DF^q(a_{i_0})\big]^{-\frac{K(t)}{q}} \leqslant E_0(t) \leqslant D_2'\big[DF^q(a_{i_0})\big]^{-\frac{K(t)}{q}} \quad (112)$$

with

$$D_1' = D_1\big[DF^q(a_{i_0})\big]^{[\frac{2a_{\max}}{pT}]+2} \qquad D_2' = D_2\big[DF^q(a_{i_0})\big]^{\frac{1}{q}}. \quad (113)$$

We choose m_2 such that

$$0 < m_2 < c_4^{-\frac{1}{2}} D_1'^{\frac{1}{4}}\big[DF^q(a_{i_0})\big]^{\frac{1}{4q}} \quad (114)$$

and we denote

$$C_1 = \left(\sqrt{D_1'} - c_4 m_2^2\big[DF^q(a_{i_0})\big]^{-\frac{1}{2q}}\right)^2 \qquad C_2 = \left(\sqrt{D_2'} + c_4 m_2^2\big[DF^q(a_{i_0})\big]^{-\frac{1}{2q}}\right)^2. \quad (115)$$

Then, using equation (99)

$$C_1\big[DF^q(a_{i_0})\big]^{-\frac{K(t)}{q}} \leqslant \tilde{E}(t) \leqslant C_2\big[DF^q(a_{i_0})\big]^{-\frac{K(t)}{q}} \quad (116)$$

for any $t \geqslant 0$ and $0 \leqslant m \leqslant \min(m_1, m_2)$.

Let us relate $K(t)$ to t. For $t \geqslant 0$

$$t + a_{\max} \geqslant a(0) = F(-a(0))$$

and therefore $K(t) \geqslant 1$. For any $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$ the relation

$$-\frac{T}{n} < \frac{F^n(x) - x}{n} - \rho(F) < \frac{T}{n} \tag{117}$$

holds by proposition II.2.3 of [10] (the same relation was used in the proof of lemma 2.17 in [5]). Putting here $n = K(t)$, $x = F^{-n}(t + a_{\max})$, $\rho(F) = \frac{p}{q}T$ and taking into account definitions (58) and (59), we obtain

$$\frac{t}{pT} + \frac{a_{\max} - a(0) - T}{pT} < \frac{K(t)}{q} < \frac{t}{pT} + \frac{a_{\max} + a(0) + T}{pT}. \tag{118}$$

Now equation (116) gives

$$C_1' \mathrm{e}^{\gamma t} \leqslant \tilde{E}(t) \leqslant C_2' \mathrm{e}^{\gamma t} \tag{119}$$

with

$$C_1' = C_1 \big[DF^q\big(a_{i_0}\big)\big]^{\frac{T + a(0) - a_{\max}}{pT}} \qquad C_2' = C_2 \big[DF^q\big(a_{i_0}\big)\big]^{-\frac{T + a(0) + a_{\max}}{pT}}. \tag{120}$$

Let us estimate the contribution of the mass term to the energy (14). Using the estimate (51),

$$\int_0^{a(t)} \tilde{\varphi}(\xi, \eta)^2 \, \mathrm{d}x \leqslant c^2 a_{\max} \, \mathrm{e}^{a_{\max} m^2 (t + a_{\max})}. \tag{121}$$

If we now denote

$$m_0 = \min\left(m_1, m_2, \sqrt{\frac{\gamma}{a_{\max}}}\right) \tag{122}$$

$$A = C_1' \qquad B = C_2' + \tfrac{1}{2} c^2 a_{\max} m_0^2 \, \mathrm{e}^{a_{\max}^2 m_0^2} \tag{123}$$

the estimate (85) follows for every $0 \leqslant m \leqslant m_0$ and $t \geqslant 0$. □

Remark. If the condition $m \leqslant \sqrt{\frac{\gamma}{a_{\max}}}$ is relaxed from equation (122) we still have exponential lower and upper bounds for the energy time-development but the upper exponent may be higher than that for the massless case. However, we cannot claim that such a bound would be saturated as we do not know whether the estimate (51) can be improved substantially. In particular, we do not know whether the field φ is bounded as in the massless case since we have only been able to prove the exponential estimate.

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